Deriving Solutions to the Differential Equations Describing an Object Falling Through an Atmosphere in a Gravitational Field

We shall here derive equations for the velocity v and acceleration a with respect to time t for an object falling vertically under gravity with weight $W = mg$, and experiencing drag proportional to its velocity $D \propto -v$ by some factor b (D is negative because it acts opposite the direction of motion).

As it turns out (but which won't be derived here), that drag coefficient is $b = (C_d \rho v A)/2$ where C_d - also called the drag coefficent - is a measure of the object's aerodynamic quality, ρ is the air density, v is the velocity of the object (which means that technically $D \propto -v^2$), and A is the total surface area of the object in contact with the resistive force of the air. For simplicity however, we will just use the formula $D = -bv$ for drag.

From Newton's Second Law, we know that the net force F_{net} acting on the object is clearly just

$$mg - bv = ma. \tag{1}$$

We may find an expression for v by rearanging (1):

$$v = \frac{m(g-a)}{b}. \tag{2}$$

This is actually done in order to find the terminal velocity v_T, the special case for v when $a = 0$:

$$v_T = \frac{mg}{b}. \tag{3}$$

In other words, v_T is the velocity v that satisfies the equilibrium condition

$$\frac{|W|}{|D|} = \frac{mg}{bv} = 1.$$

Let's now rewrite (1), but without any acceleration terms, so replacing a with $\frac{dv}{dt}$:

$$mg - bv = m\frac{dv}{dt},$$

and as g is also an acceleration term, we can divide the whole thing by b to get

$$v_T - v = \frac{m}{b}\frac{dv}{dt},$$

giving us v_T in place of mg, following relation (3). Just because by the end we will want a linear ressult for convenience, it is better to swap v_T and v before things get too complicated:

$$v - v_T = -\frac{m}{b}\frac{dv}{dt}.$$

As we want to isolate the velocity terms on the LHS and the time term on the RHS - hence giving us of course v as a function of t: $v(t)$ - we can simply invert our equation and shift the dv term over to the LHS as follows:

$$\begin{aligned}\frac{1}{v - v_T} &= -\frac{b}{m}\frac{dt}{dv},\\ \frac{dv}{v - v_T} &= -\frac{b}{m}dt.\end{aligned}$$

However this still leaves us with the derivative terms dv and dt, so we will need to take definite integrals with respect to v and t:

$$\begin{aligned}\int_0^v \frac{dv}{v - v_t} &= -\frac{b}{m}\int_0^t dt,\\ ln|v - v_T| - ln|0 - v_T| &= -\frac{b}{m}t,\\ ln\left|\frac{v - v_T}{-v_T}\right| &= -\frac{b}{m}t.\end{aligned}$$

To get rid of this logarithm on the LHS, we can take its exponential function

$$\begin{aligned}exp\left(ln\left|\frac{v - v_T}{-v_T}\right|\right) &= exp\left(-\frac{b}{m}t\right),\\ \frac{v - v_T}{-v_T} &= e^{-\frac{b}{m}t},\end{aligned}$$

then to allow for the isolation of v, we rewrite the remaining fraction as

$$1 - \frac{v}{v_T} = e^{-\frac{b}{m}t}.$$

Now for the final step, upon fully isolating v by shifting everything else over to the RHS, v may be crowned $v(t)$:

$$\boxed{v(t) = v_T\left(1 - e^{-\frac{b}{m}t}\right)} \tag{4}$$

The second goal is to derive an equation for $a(t)$, which, now that we have our first solution (4), is relatively straightforward. I will just make some adjustments to eq(4) first, replacing v_T with mg/b by relation (3), and expanding the RHS so that the product rule will not be necessary:

$$v(t) = \frac{mg}{b} - \frac{mg}{b}e^{-\frac{b}{m}t}.$$

Then, it's a simple matter of taking the time derivative

$$\dot{v}(t) = \frac{d}{dt}\left(\frac{mg}{b} - \frac{mg}{b}e^{-\frac{b}{m}t}\right),$$

which is equivalent to $a(t)$. So this yields

$$a(t) = -\frac{mg}{b}e^{-\frac{b}{m}t}\left(-\frac{b}{m}\right),$$

since $de^u = e^u du$. After cancelling some terms, we arrive at our second solution:

$$\boxed{a(t) = ge^{-\frac{b}{m}t}} \tag{5}$$

Notice that solutions (4) and (5) are of the form $k\left(1 - e^{-\frac{t}{\tau}}\right)$ and $ke^{-\frac{t}{\tau}}$ respectively. These are the general forms (of $v(t)$ and $a(t)$) for a situation involving conflict between a constant and a variable force.