1 Equations of Motion

1.1 Derivation

Given that acceleration relates to position with $a(t) = \frac{d^2x}{dt^2}$ with respect to time, we can find an expression for x in terms of a and t by taking the double integral of $a(t)$ with respect to t,

$$\int \int_{\mathbb{R}} a(t)\ d\tau dt = \frac{1}{2}at^2 + bt + c,$$

using τ as a dummy variable. $\frac{1}{2}at^2$ behaves like $\frac{d^2x}{dt^2}$ over all time (acceleration), balanced by a factor $\frac{1}{2}t^2$; bt behaves like $\frac{dx}{dt}$ at initial time (initial velocity), balanced by a factor t; and c behaves like x at initial time (initial position). We can now rewrite the equation as

$$x(t) = x_0 + ut + \frac{1}{2}at^2. \tag{1.1.1}$$

Given this, and since $v(t) = \frac{dx}{dt}$, we can derive a similar expression for velocity v in terms of a and t,

$$\frac{d}{dt}(x_0 + ut + \frac{1}{2}at^2) = u + at,$$

and we can write this as

$$v(t) = u + at. \tag{1.1.2}$$

From (1.1.2) we can see that

$$t = \frac{v - u}{a},$$

and if we plug this into (1.1.1) we get

$$v^2 = u^2 + 2a(x - x_0)$$

which gives us a relation between x, v and a, but without requiring t. It is commonly written as

$$v^2 = u^2 + 2as. \tag{1.1.3}$$

Where s is the displacement. These are the three primary equations of motion for a particle experiencing constant acceleration.

1.2 Projectile Motion in 1 Dimension

We can apply (1.1.1) to the scenario of an object launched vertically with some initial velocity u and negative acceleration g due to the force of gravity (but with no horizontal motion). We take the origin to be $0m$ above the ground:

$$y(t) = ut - \frac{1}{2}gt^2. \tag{1.2.1}$$

From this equation it is easy to deduce anything about the object's motion over time. For example, let's denote t^* to be any time the object is on the ground:

$$\begin{aligned} y(t^*) = ut^* - \frac{1}{2}gt^{*2} &= 0, \\ \left(u - \frac{1}{2}gt^*\right)t^* &= 0, \\ t^* = \frac{2u}{g}, \text{ and } t^* &= 0. \end{aligned} \tag{1.2.2}$$

As expected, it is on the ground initially and at one other time, when it lands again. It should be obvious that the object takes equally as long to go up as to come down, as the acceleration $-g$ remains constant throughout. Therefore, if we define T $(= \frac{1}{2}t^*)$ to be the time taken to reach maximum height, the maximum height Y itself can be found:

$$\begin{aligned} Y = y(T) &= u\left(\frac{u}{g}\right) - \frac{1}{2}g\left(\frac{u}{g}\right)^2, \\ &= \frac{u^2}{g} - \frac{1}{2}\frac{u^2}{g}, \\ Y &= \frac{u^2}{2g}. \end{aligned} \tag{1.2.3}$$

We have now derived the equations of all non-implicit critical times and positions for this object.

1.3 Mathematical Interlude: Vector Algebra

Before we move on to higher dimensions we need to establish a rudementary understanding of vecrtor algebra. Let $\vec{A}$ denote a vector in a 2-dim vector space. A vector in n-dimensions is made up of n components:

$$\vec{A} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}}, \tag{1.3.1}$$

where $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are unit vectors (vectors of unit length along the x and y axes respectively), and a_x and a_y are multiples of the magnitude of a unit vector. Together, these terms are called the 'vector components'. (We may also write a vector component as $\vec{a}_{x,y}$.) The magnitude of $\vec{A}$ itself (which can be written as either $|\vec{A}|$, or simply as A when the context is understood) is found with the usual method for finding the length of a line segment,

$$|\vec{A}| = \sqrt{{a_x}^2 + {a_y}^2}. \tag{1.3.2}$$

The sum of vectors is also a vector:

$$\vec{A} + \vec{B} = \vec{C}. \tag{1.3.3a}$$

The components of $\vec{C}$ are found by adding the components of $\vec{A}$ and $\vec{B}$:

$$\vec{A}+\vec{B}=(a_x+b_x)\hat{\mathbf{i}}+(a_y+b_y)\hat{\mathbf{j}}=c_x\hat{\mathbf{i}}+c_y\hat{\mathbf{j}}=\vec{C}. \tag{1.3.3b}$$

It is not necessary to provide any further details about vector algebra at this time, but we shall revisit the topic in due course.

1.4 Projectile Motion in 2 Dimensions

When we consider an object moving in more than 1 dimension, its path can be described using vectors; the horizontal motion is independant of the verticle motion so the overall motion can be viewed as sum of the paths over both planes. In other words, the motion along a path can be represented by vector components. Technically, even motion in 1 dimension can be described using vectors, but this is unnecessary as 1-dim vectors possess only 1 component, i.e. it is essentially just a scalar.

To talk about projectie motion in 2 dimensions, we can still use equation (1.2.1) for the verticle path (with a slight adjustment) and, since there is no acceleration along the horizontal path, the simple $\vec{x}(t)=\vec{v}_x t$ where $\vec{v}_x=\vec{u}_x$, will do (again with a slight adjustment). The adjustments we will make have to do with explicitly writing the vector components of $\vec{u}$, which are simply

$$\vec{u}_x=\vec{u}cos\theta \text{ and } \vec{u}_y=\vec{u}sin\theta,$$

θ being of course the launch angle. Hence, the explicit forms of the equations of motion are:

$$\vec{x}(t) = \vec{u}cos\theta t, \tag{1.4.1}$$

$$\vec{y}(t) = \vec{u}sin\theta t - \frac{1}{2}gt^2, \tag{1.4.2}$$

if both $y(t^*)$ are equivalent. However we must not forget to add the term y_0 to (1.4.2) if our projectile was launched from some altitude, for example from a tall building, and eventuallly lands on the ground. Here we can take ground level to be $0m$ and the height of the building to be y_0.